数理解析研究所講究録
660巻 1988年 180-196

Arithmetic of Real Principal Symbols of Regular Holonomic Microfunctions*

Masakazu Muro 室 政和
Kochi University 高知大学理学部

(* This research is partially supported by Grant-in-Aid for Scientific Research 62740093 of Ministry of Education in Japan.)

Contents.

Introduction.
§1. Microdifferential operators of logarithmic order.
§2. Holonomic systems and their hyperfunction solutions.
§3. Regular holonomic microfunctions and their real principal symbols.
References.

Introduction.

The purpose of this note is to give the definition of real principal symbols for microfunction solutions of a regular holonomic system and study their fundamental properties. Arithmetic of real principal symbols helps explicit calculation of regular holonomic microfunctions especially when we have to determine all the hyperfunction solutions of a regular holonomic system supported in low-dimensional subvarieties. There are many applications of real principal symbols. The detailed theory and applications will be given in the future paper that the author is preparing.

Hyperfunctions were originally introduced as boundary values of holomorphic functions by M.Sato in terms of local cohomology. It is often difficult to express a concrete hyperfunction using a set of boundary values of holomorphic functions except for some cases of one variable function. We have already known the method to deal with a hyperfunction using its plane wave expansion in order to overcome such difficulty. However, while this method is powerful when we treat the hyperfunctions in a small microlocal area, it may be helpless when we analyze them in a global domain. On the other hand, the hyperfunctions that we want to calculate concretely are often solutions of systems of linear differential equations called regular holonomic systems (see §3), for example, group-invariant hyperfunctions appearing in the harmonic analysis on homogeneous spaces. See [Muro2].

We shall give a new method to calculate hyperfunctions by restricting our attention to hyperfunction solutions of regular holonomic systems. This method is computing "real principal symbols" of the hyperfunctions instead of dealing with the hyperfunctions themselves. The real principal symbols are defined based on the fact that microfunction solutions of regular holonomic systems are obtained by multiplying microdifferential operators of logarithmic order, which is defined in §1, to a delta function. In this note we shall give the definition of microfunction solutions of regular holonomic systems, which are called regular holonomic microfunctions, and state their properties in §3.

The real principal symbol was originally introduced by

[Kashiwara1] for microfunction solutions of simple holonomic systems. Our definition of the real principal symbols of regular holonomic microfunctions is a natural generalization of Kashiwara's definition. Many properties of real principal symbols of simple microfunctions are valid for those of regular holonomic microfunctions. The author thinks that real principal symbols are more easily manipulated than the plane wave expansion. Moreover, a regular holonomic hyperfunction is completely determined by the real principal symbols on "generic" points of its support as a microfunction. This follows immediately from a general theorem stated in §2.

Notations. $\mathbb{Z}$: the set of integers. $\mathbb{Z}_{\geq 0}$: the set of non-negative integers. $\mathbb{R}$: the set of real numbers. $\mathbb{C}$: the set of complex numbers.

§1. Microdifferential operators of logarithmic order.

We shall in this section give the definition of a microdifferential operator of logarithmic order and state some fundamental properties of it without detailed explanation. All the properties are parallel to those of microdifferential operators of fractional order defined in [Sato-Kashiwara-Kimura-Oshima].

Let X be a complex manifold of dimension n and let T^*X be its cotangent bundle. For an arbitrary fixed point $(z_0,\xi_0)\in T^*X$, we take a local coordinate $(z_1,\ldots,z_n;\xi_1,\ldots,\xi_n)$ of T^*X near (z_0,ξ_0). Let λ be a complex number and let m be a non-negative integer. We would like to define the sheaf of *microdifferential operators of fractional order* λ *and logarithmic order* m, which is denoted by $\mathcal{E}_X(\lambda,m)$. A local section of $\mathcal{E}_X(\lambda,m)$ near (z_0,ξ_0) is given by a formal infinite sum of holomorphic functions defined near (z_0,ξ_0): it is presented as

$$(1.1)\qquad P(z,D_z)=\sum_{j=0}^{\infty} P_{\lambda-j}(z,D_z),$$

where $P_{\lambda-j}(z,\xi)$ is a holomorphic function defined in a conic neighborhood Ω of (z_0,ξ_0) satisfying the following two conditions:

$$(1.2)\quad \text{i)}\ (\sum_{i=1}^{n} \xi_i\frac{\partial}{\partial\xi_i}-(\lambda-j))^{m+1}\ P_{\lambda-j}(z,\xi) = 0,$$

ii) For any compact set K in Ω and $\varepsilon>0$, there exists a constant C_K such that $\sup_K|P_{\lambda-j}(z,\xi)|\le(-j)!\cdot C_K{}^{-j}$ for $j>0$.

We denote by $\mathcal{E}_X(\lambda,m)_{(z_0,\xi_0)}$ the set of local sections of $\mathcal{E}_X(\lambda,m)$ at (z_0,ξ_0). It has naturally a structure of $\mathbb{C}$-module. We can define the product of two local sections in the following way. Let $P(z,D_z)\in\mathcal{E}_X(\lambda,m)_{(z_0,\xi_0)}$ and $Q(z,D_z)\in\mathcal{E}_X(\mu,\ell)_{(z_0,\xi_0)}$. We define the product $R(z,D_z):=P(z,D_z)Q(z,D_z)\in\mathcal{E}_X(\lambda+\mu,m+\ell)_{(z_0,\xi_0)}$ by

$$(1.3)\quad R(z,D_z) = \sum_{\ell=0}^{\infty} R_{\lambda+\mu-\ell}(z,D_z) \quad \text{with}$$
$$R_{\lambda+\mu-\ell}(z,\xi):= \sum_{\ell=j+k+|\alpha|}\frac{1}{\alpha!}((\frac{\partial}{\partial\xi})^{\alpha}\ P_{\lambda-j}(z,\xi))((\frac{\partial}{\partial z})^{\alpha}\ Q_{\mu-k}(z,\xi)).$$

This product is non-commutative. It satisfies the associative law :

$(PQ)S=P(QS)$ where P,Q and S are microdifferential operators of logarithmic order, and the distributive law : $P(Q+S)=PQ+PS$ where $P,Q,$ and S are microdifferential operators of logarithmic order and the fractional orders of Q and S are the same.

The above definition of $\mathcal{E}_X(\lambda,m)_{(x_0,\xi_0)}$ depends on the choice of local coordinate systems. However, we may define the sheaf $\mathcal{E}_X(\lambda,m)$ free from the coordinate by putting the transition law from the old coordinate to the new coordinate in the following way. Let $z^{\sim}:=(z_1^{\sim},\ldots,z_n^{\sim})$ be a new local coordinate system and let $(z^{\sim},\xi^{\sim}):=(z_1^{\sim},\ldots,z_n^{\sim};\xi_1^{\sim},\ldots,\xi_n^{\sim})$ be the corresponding local coordinate system of T^*X, i.e., $\xi_j^{\sim}=\sum_k\frac{\partial z_k}{\partial z_j^{\sim}}\cdot\xi_k$. Then the microdifferential operator $P(z,D_z)=\sum_{j=0}^{\infty} P_{\lambda-j}(z,D_z)$ written in the old coordinate (z,ξ) is transformed to the one written in new coordinate $\sum_{\ell=0}^{\infty} P_{\lambda-\ell}^{\sim}(z^{\sim},D_{z^{\sim}})$ with

$$\text{(1.4)}\quad P_{\lambda-\ell}^{\sim}(z^{\sim},\xi^{\sim}) =\sum \frac{1}{\nu!\alpha_1!\cdots\alpha_\nu!}\langle\xi^{\sim},(\frac{\partial}{\partial z})^{\alpha_1}z^{\sim}\rangle\cdots\langle\xi^{\sim},(\frac{\partial}{\partial z})^{\alpha_\nu}z^{\sim}\rangle(\frac{\partial}{\partial\xi})^{\alpha} P_{\lambda-j}(z,\xi).$$

Here, the indices of $\sum$ in (1.4) run over

$$\text{(1.5)}\quad \begin{aligned} &j\in\mathbb{Z}_{\geq0},\ \nu\in\mathbb{Z}_{\geq0},\ (\alpha_1,\cdots,\alpha_\nu)\in(\mathbb{Z}_{\geq0}{}^n)^\nu.\\ &\alpha=\alpha_1+\cdots+\alpha_\nu \text{ with } |\alpha_1|,\cdots,|\alpha_\nu|\geq2,\\ &\ell=j-\nu+|\alpha_1|+\cdots+|\alpha_\nu|>0.\end{aligned}$$

and we put $\langle\xi^{\sim},(\frac{\partial}{\partial z})^{\beta}z^{\sim}\rangle:=\sum_{j=1}^{n}\xi_j^{\sim}\cdot(\frac{\partial}{\partial z})^{\beta}z_j^{\sim}$ for $B\in\mathbb{Z}_{\geq0}{}^n$.

Thus we define the set $\mathcal{E}_X(\lambda,m)_{(z_0,\zeta_0)}$ of local sections of microdifferential operators of fractional order λ and logarithmic order m independently from the choice of local coordinate systems at (z_0,ζ_0). The sheaf $\mathcal{E}_X(\lambda,m)$ on T^*X is obtained as the sheaf whose stalk at $(z_0,\zeta_0)\in T^*X$ is $\mathcal{E}_X(\lambda,m)_{(z_0,\zeta_0)}$, which is also defined intrinsically. We call $\mathcal{E}_X(\lambda,m)$ the sheaf of *microdifferential operators of fractional order* λ *and of logarithmic order* m. The product of two global sections of $\mathcal{E}_X(\lambda,m)$ is computed from the formula (1.3). When $\zeta_0=0$, the stalk $\mathcal{E}_X(\lambda,m)_{(z_0,0)}$ coincides with the sheaf of differential operators of order λ with holomorphic coefficients if λ is a non-negative integer.

Concluding this section, we shall define the principal symbol of a section of $\mathcal{E}_X(\lambda,m)$, which is a section of the sheaf of holomorphic functions $\mathcal{O}_{T^*X}$. Let $P(z,D_z)=\sum_{j=0}^{\infty}P_{\lambda-j}(z,D_z)$ be a section of $\mathcal{E}_X(\lambda,m)$. The "highest" order term $P_\lambda(z,D_z)$ is called the *principal part* of $P(z,D_z)$ and the symbol function $P_\lambda(z,\zeta)$ is called the *principal symbol* of $P(z,D_z)$. We denote by $\sigma(P)(z,\zeta)$ the principal symbol of $P(z,D_z)$. They are free from the choice of local coordinate systems. When $P(z,D_z)$ is a section of $\mathcal{E}_X(\lambda,m)$ and $Q(z,D_z)$ is a section of $\mathcal{E}_X(\mu,\ell)$, the principal symbol of the product $PQ\in\mathcal{E}_X(\lambda+\mu,m+\ell)$ is $\sigma(P)\sigma(Q)$.

Let M be a real analytic manifold whose complexification is X. Then $\mathcal{E}_X(\lambda,m)|_{T^*M}$ is a subsheaf of the sheaf of micro-local operators on T^*M and naturally acts on the sheaf of microfunctions on T^*M. In §3 we will state that any microfunction solution of a regular holonomic system is obtained by multiplying a microdifferential operators to a delta-function if it is defined near a "generic" point, i.e., a point which is regular and in regular position. (For definition, see §3.) Moreover we will see that for a fixed holonomic system, microfunction solutions are completely determined by the principal symbols of microdifferential operators acting to the delta-functions. This is an important fact when we will deal with a solution of a regular holonomic system.

§2. Holonomic systems and their hyperfunction solutions.

We put $\mathcal{E}_X := \bigcup_{\lambda\in\mathbb{Z}} \mathcal{E}_X(\lambda,0)$, which is the sheaf of ordinary microdifferential operators on T^*X. Let π be the projection map from T^*X to X. The direct image $\pi_*(\mathcal{E}_X)$ is isomorphic to the sheaf $\mathcal{D}_X$ of differential operators on X with holomorphic coefficients.

Let $\mathfrak{M}$ be a holonomic system on X. That is, $\mathfrak{M}$ is a left coherent $\mathcal{D}_X$-module whose characteristic variety $\mathrm{ch}(\mathfrak{M}) := \mathrm{supp}(\mathcal{E}_X \otimes_{\pi^{-1}(\mathcal{D}_X)} \pi^{-1}(\mathfrak{M}))$ is a Lagrangian subvariety. For the details about holonomic systems see [Kashiwara2] and [Kashiwara-Kawai]. We denote by $\bigcup_{i\in I}\Lambda_{i\mathbb{C}}$ the irreducible component decomposition of $\mathrm{ch}(\mathfrak{M})$. We put $X_i := \pi(\Lambda_{i\mathbb{C}})$. We denote by $X_{i\ \mathrm{reg}}$ the set of non-singular points of X_i. Then we have $\Lambda_i = \overline{T^*_{X_{i\ \mathrm{reg}}} X}$. Here $T^*_A X$ means the conormal bundle of A in X and $\overline{}$ stands for the closure. We say that $(z,\zeta)\in\Lambda_i$ is *in regular position* with respect to $\Lambda_{i\mathbb{C}}$ if $z\in X_{i\ \mathrm{reg}}$. We denote by $\overset{\circ}{\Lambda}_{i\mathbb{C}}$ the set of points in $\Lambda_{i\mathbb{C}}$ in regular position with respect to $\Lambda_{i\mathbb{C}}$. We say that $p\in\mathrm{ch}(\mathfrak{M})$ is a *regular* point in $\mathrm{ch}(\mathfrak{M})$ if it is a non-singular point of $\mathrm{ch}(\mathfrak{M})$. We denote by $\mathrm{ch}(\mathfrak{M})_{\mathrm{reg}}$ the set of regular points of $\mathrm{ch}(\mathfrak{M})$. The sets $\mathrm{ch}(\mathfrak{M})$ and $\bigcup_{i\in I}\mathring{\Lambda}_{i\mathbb{C}}$ are open dense subset in $\mathrm{ch}(\mathfrak{M})$ and so is $\mathrm{ch}(\mathfrak{M})_{\mathrm{reg}}\cap(\bigcup_{i\in I}\mathring{\Lambda}_{i\mathbb{C}})$. The complement of the set $\mathrm{ch}(\mathfrak{M})_{\mathrm{reg}}\cap(\bigcup_{i\in I}\mathring{\Lambda}_{i\mathbb{C}})$ in $\mathrm{ch}(\mathfrak{M})$ is an analytic subset of codimension larger than one.

Let M be a real form of X. We would like to study a hyperfunction solution on M of $\mathfrak{M}$ by resolving it to the microfunction on T^*M. The reason why we have considered the set $\mathrm{ch}(\mathfrak{M})_{\mathrm{reg}}\cap(\bigcup_{i\in I}\mathring{\Lambda}_{i\mathbb{C}})$ is for two hyperfunction solutions which coincide on the set $\mathrm{ch}(\mathfrak{M})_{\mathrm{reg}}\cap(\bigcup_{i\in I}\mathring{\Lambda}_{i\mathbb{C}})$ as microfunctions actually coincide as hyperfunctions on M. We shall explain about it more precisely. Let $\mathscr{B}_M$ be the sheaf of hyperfunctions on M and let $\mathscr{C}_M$ be the sheaf of microfunctions on T^*M ($\simeq T^*_MX$). The spectral map $sp:\mathscr{B}_M \longrightarrow \pi_*(\mathscr{C}_M)$ gives an isomorphism between them. Here, π stands for the projection map $T^*M \longrightarrow M$. The sheaf $\mathscr{B}_M$ is naturally a $\mathscr{D}_X|_M$-module. A hyperfunction solution of $\mathfrak{M}$ is defined to be a section of $\mathscr{H}om_{\mathscr{D}_X}(\mathfrak{M},\mathscr{B}_M)$ in our situation (see [Kashiwara3]). Let F_1 and F_2 be two sections of $\mathscr{H}om_{\mathscr{D}_X}(\mathfrak{M},\mathscr{B}_M)$ and let u be a section of $\mathfrak{M}$. Then $F_1(u)$ and $F_2(u)$ are sections of hyperfunctions whose singular spectra are contained in $\mathrm{ch}(\mathfrak{M})_{\mathbb{R}}:=\mathrm{ch}(\mathfrak{M})\cap T^*M$.

Theorem 2.1 ([Muro1])

Let $A_{\mathbb{C}}$ *be an analytic subset in* $\mathrm{ch}(\mathfrak{M})$ *of complex codimension larger than one. Let* $A_{\mathbb{R}}:=A_{\mathbb{C}}\cap T^*M$ *and let* F_1,F_2 *be two sections of* $\mathscr{H}om_{\mathscr{D}_X}(\mathfrak{M},\mathscr{B}_M)$. *If* $sp(F_1(u))=sp(F_2(u))$ *on* $\mathrm{ch}(\mathfrak{M})_{\mathbb{R}}-A_{\mathbb{R}}$ *for every section* u *of* $\mathfrak{M}$, *then* $F_1=F_2$.

We may set $A_{\mathbb{C}}=ch(\mathfrak{M})-(ch(\mathfrak{M})_{reg}\cap(\bigcup_{i\in I}\overset{\circ}{\Lambda_{i\mathbb{C}}}))$ in this theorem. Then a hyperfunction solution of a holonomic system is completely determined by the data on $ch(\mathfrak{M})_{reg}\cap(\bigcup_{i\in I}\overset{\circ}{\Lambda_{i\mathbb{C}}})\cap T^*M$. In the next section we shall suppose that $\mathfrak{M}$ is a "regular" holonomic system and define the real principal symbol of a microfunction solution of $\mathfrak{M}$. Two locally-defined microfunction solutions with the same real principal symbol coincide with each other. However, note that we can define the real principal symbol of a microfunction solution only at regular points in regular position. Theorem 2.1 guarantees that it is sufficient to compute the real principal symbols on regular points of $ch(\mathfrak{M})$ in order to determine the hyperfunction solution. This theorem will be practically used for the proof of Theorem 3.5 and Corollary 3.6.

§3. Regular holonomic microfunctions and their real principal symbols.

We have proved that the data of microfunction solutions at regular points determine the corresponding hyperfunction solution if it exists (Theorem 2.1). However, it is not easy to express a microfunction in a form to easily manipulate. We shall in this section introduce real principal symbols of regular holonomic microfunctions and state their fundamental properties. Real principal symbols are helpful to express microfunctions explicitly.

We begin with the review of the definition of regular holonomic system following to [Kashiwara2] and [Kashiwara-Kawai]. We set

$$\mathcal{L}_{\lambda,m}:=\mathcal{E}_X/\mathcal{E}_X(x_1D_{x_1}-\lambda)^m+\mathcal{E}_Xx_2+\cdots+\mathcal{E}_Xx_\ell+\mathcal{E}_XD_{x_{\ell+1}}+\cdots+\mathcal{E}_XD_{x_n},$$

with a complex number λ and an integer m. Then $\mathcal{L}_{\lambda,m}$ is a holonomic $\mathcal{E}_X$-module defined near $p=(0,dx_1)\in T^*X$. This is a holonomic system supported in $\Lambda:=\{(x,\xi)\in T^*X;\ x_1=\cdots=x_\ell=0,\xi_{\ell+1}=\cdots=\xi_m=0\}$. Let $\mathfrak{M}$ be a holonomic $\mathcal{D}_X$-module on X. We say that $\mathfrak{M}$ is *regular holonomic* at a point $p\in ch(\mathfrak{M})$ if $ch(\mathfrak{M})$ is non-singular near p and $\mathfrak{M}^{\mathcal{E}}:=(\mathcal{E}_X\otimes\mathfrak{M})$ is isomorphic to a direct sum $\oplus_j\mathcal{L}_{\lambda_j,m_j}$ through a quantized contact transformation. In particular, we call $\mathfrak{M}$ a regular holonomic system if $\mathfrak{M}$ is regular holonomic at any point p in $ch(\mathfrak{M})_{reg}$.

Let $\mathfrak{M}$ be a regular holonomic system on X. Let $\bigcup_{i\in I}\Lambda_{i\mathbb{C}}=ch(\mathfrak{M})$ be an irreducible component decomposition of $ch(\mathfrak{M})$. Let M be a real form of X. The real locus $\Lambda_{i\mathbb{R}}:=\Lambda_{i\mathbb{C}}\cap T^*M$ is not always real Lagrangian subvariety in T^*M. Henceforth, we suppose that

(3.1) each real locus $\Lambda_{i\mathbb{R}}$ is a real Lagrangian subvariety in T^*M or a variety of dimension zero.

We let $\overset{\circ}{\Lambda}_{i\mathbb{C}}$ be the set of points in regular position with respect to $\Lambda_{i\mathbb{C}}$ and let $\overset{\circ}{\Lambda}_{i\mathbb{R}}:=\overset{\circ}{\Lambda}_{i\mathbb{C}}\cap T^*M$. The real locus $ch(\mathfrak{M})\cap T^*M$ (resp. $ch(\mathfrak{M})_{reg}\cap T^*M$) is denoted by $ch(\mathfrak{M})_{\mathbb{R}}$ (resp. $ch(\mathfrak{M})_{reg\mathbb{R}}$).

Now we go to the definition of the real principal symbol of a microfunction solution of $\mathfrak{M}$. We denote by $\mathfrak{M}^{\mathcal{E}}$ the $\mathcal{E}_X$-module $\mathcal{E}_X\otimes_{\pi^{-1}(\mathcal{D}_X)}\pi^{-1}(\mathfrak{M})$ on T^*M. Let $p=(x_0,y_0)$ be a point and let F be a local section of $\mathcal{H}om_{\mathcal{E}_X}(\mathfrak{M}^{\mathcal{E}},\mathcal{C})$ defined near p. Such a local section F is interpreted as a microfunction solution of $\mathfrak{M}^{\mathcal{E}}$ from our view point. For a local section u of $\mathfrak{M}^{\mathcal{E}}$, F(u) is a section of microfunction defined near p. We can define the real principal symbol of F(u) if p is contained in $ch(\mathfrak{M})_{reg\ \mathbb{R}}$.

First we suppose that p is contained in the set $ch(\mathfrak{M})_{reg\ \mathbb{R}} \cap (\bigcup_{i\in I}\Lambda^{\circ}_{i\mathbb{R}})$, i.e., p is a regular point in $ch(\mathfrak{M})$ in regular position. From the condition (3.1), p is contained in a smooth real Lagrangian subvariety $\Lambda_{i\mathbb{R}}$ in T^*M. In fact, if $\Lambda^{\circ}_{i\mathbb{R}}$ is of dimension zero, it is contained in the zero section $T^*_M M$. Thus it is not contained in $ch(\mathfrak{M})_{reg\ \mathbb{R}}$. Since $p\in\Lambda^{\circ}_{i\mathbb{C}}$ is in regular position with respect to $\Lambda_{i\mathbb{C}}$, $\Lambda_{i\mathbb{C}}$ is given as the conormal bundle of $\pi_*(\Lambda_{i\mathbb{C}})$. In the real form T^*M, the real locus $\Lambda^{\circ}_{i\mathbb{R}}$ is given by the real conormal bundle $T^*_{\pi_*(\Lambda_{i\mathbb{R}})}M$ near p. The subvariety $\pi(\Lambda^{\circ}_{i\mathbb{R}})$ is a non-singular subvariety near $x_0\in M$, it is written as

$$(3.1)' \quad \pi(\Lambda^{\circ}_{i\mathbb{R}})=\{(x_1,\cdots,x_n)\in M;\ x_1=x_2=\cdots=x_{\ell}=0\},$$

by using a suitable local coordinate $(x_1,\cdots,x_n)$ near x_0. Then $\Lambda^{\circ}_{i\mathbb{R}}=\{(x,\xi)\in T^*M; x_1=\cdots=x_{\ell}=0,\xi_{\ell+1}=\cdots=\xi_n=0\}$ and $(\xi_1,\cdots,\xi_{\ell},x_{\ell+1},\cdots,x_n)$ forms a local coordinate system on $\Lambda^{\circ}_{i\mathbb{R}}$ near p.

Proposition 3.1. *Let* F(u) *be a microfunction defined near* p *in* $\Lambda^{\circ}_{i\mathbb{R}}$ *with* F *a section of* $\mathcal{H}om_{\mathcal{E}_X}(\mathfrak{M}^{\mathcal{E}},\mathcal{C})$ *and* u *a section of* $\mathfrak{M}^{\mathcal{E}}$. *Take a local coordinate* $(x_1,\cdots,x_n)$ *defined in* (3.1)′. *Then* F(u) *is written as*

(3.2) $\quad F(u)=(\oplus_k P_k(x'',D_{x''}))\delta(x_1)\cdots\delta(x_\ell),$

where $x'=(x_1,\cdots,x_\ell)$, $x''=(x_{\ell+1},\cdots,x_n)$, $D_{x'}=(D_{x_1},\cdots,D_{x_\ell})$ *and* $D_{x''}=(D_{x_{\ell+1}},\cdots,D_{x_n})$; *each* $P_k(x'',D_{x''})$ *is a microdifferential operator of logarithmic order depending only on* x'' *and* $D_{x''}$; $\delta(t)$ *is the delta-function of one variable* t.

We denote by $\oplus_k$ the formal finite sum of microdifferential operators P_k of logarithmic order. This expression (3.2) is uniquely determined.

Remark. A sum of several microdifferential operators $\sum_k P_k(x,D_x)$ is only formally defined unless their differences of the fractional orders are integers. However, the microfunction $\sum_k P_k(x'',D_{x''})\delta(x')$ is well defined. Here, we have used the symbol $\oplus$ instead of $\sum$ since we want to stress that the sum is a formal one.

Now, we may define the real principal symbol of $F(u)$ as a section of $|\Omega_{\Lambda_{i\mathbb{R}}}|^{1/2}\otimes|\Omega_M|^{-1/2}$ at each point $\overset{\circ}{\Lambda}_{i\mathbb{R}}\cap ch(\mathfrak{M})_{reg}$. Here $|\Omega_{\Lambda_{i\mathbb{R}}}|$ and $|\Omega_M|$ are the sheaves of volume elements on $\Lambda_{i\mathbb{R}}$ and M, respectively.

Definition 3.2. (A real principal symbol at regular position) Let $F(u)$ be a microfunction given in (3.2). *The real principal symbol* of $F(u)$, which is denoted by $\sigma_{\Lambda_{i\mathbb{R}}}(F(u))$ or simply by $\sigma(F(u))$, is defined by

$$(3.3)\ \sigma_{\Lambda_{i\mathbb{R}}}(F(u))=\oplus_k\ \sigma(P_k)(x'',\xi')\sqrt{|dx''\wedge d\xi'|}/\sqrt{|dx|}.$$

Here $\sigma(P_k)$'s are principal symbols of P_k defined in §1 and $dx'':=dx_{\ell+1}\wedge\cdots\wedge dx_n$, $d\xi':=d\xi_1\wedge\cdots\wedge d\xi_\ell$ and $dx:=dx_1\wedge\cdots\wedge dx_n$. The real principal symbol $\sigma_{\Lambda_{i\mathbb{R}}}(F(u))$ is a vector-valued real analytic section of $|\Omega_{\Lambda_{i\mathbb{R}}}|^{1/2}\otimes|\Omega_M|^{-1/2}$.

Remark. In particular, if $\mathfrak{M}$ is a simple holonomic system, then $\mathfrak{M}$ is a regular holonomic system. In such case the definition of the real principal symbol has already been given in [Kashiwara2], which coincides with the definition (3.2).

The real principal symbol in definition 3.2 is only defined on $\overset{\circ}{\Lambda}_{i\mathbb{R}}$. We want to extend the real principal symbol $\sigma(F(u))$ defined on $\overset{\circ}{\Lambda}_{i\mathbb{R}}$ to $\Lambda_{i\mathbb{R}}$. Such extended section may not always be continuous but is uniquely determined. It becomes a real analytic section by multiplying a locally constant section on $\overset{\circ}{\Lambda}_{i\mathbb{R}}$ of "Maslov's index" bundle. We suppose that $\Lambda_{i\mathbb{R}}$ is a real Lagrangian subvariety in T^*M. We let $\Lambda_{i\mathbb{R}}':=\Lambda_{i\mathbb{R}}\cap ch(\mathfrak{M})_{reg}$ and $N_{\mathbb{R}}$ be the subset of $\Lambda_{i\mathbb{R}}$ consisting of points in regular position with respect to $\Lambda_{i\mathbb{C}}$. Then, from the definition of a real principal symbol, $\sigma_{\Lambda_{i\mathbb{R}}}(F(u))$ is well defined and real analytic vector-valued section of $|\Omega_{\Lambda_{i\mathbb{R}}}|^{1/2}\otimes|\Omega_M|^{-1/2}$ on $\Lambda_{i\mathbb{R}}'-N_{\mathbb{R}}$. The section $\sigma_{\Lambda_{i\mathbb{R}}}(F(u))$ can not always be extended to $N_{\mathbb{R}}$ real analytically because of a possible gap between the sections on the different connected components. We have to introduce the "Maslov's index" bundle to repair the gap.

We shall explain the Maslov's index. Let p be a point in T^*M. we set $V:=T_p(T^*M)$,i.e., the tangent space of T^*M at p. There is the skew symmetric bilinear form E naturally introduced from the symplectic structure on T^*M. Namely, letting ω be the fundamental 1-form on T^*M, E is defined to be $E(\nu_1,\nu_2):=\langle d\omega,\nu_1\wedge\nu_2\rangle$ with $\nu_1,\nu_2\in V$. We say that a vector subspace λ in V is a Lagrangian plane in V if every vector in λ is orthogonal to λ and if the dimension of λ is n. We denote by LGrass(V) the totality of all Lagrangian planes in V. This is called *Lagrangian Grassmannian*, which has the natural structure of smooth real analytic variety. For three elements $\lambda_1,\lambda_2,\lambda_3\in \mathrm{LGrass}(V)$, we set $\tau(\lambda_1,\lambda_2,\lambda_3):=\mathrm{sgn}(Q)$ where Q is the quadratic form on $x_1\oplus x_2\otimes x_3\in\lambda_1\oplus\lambda_2\oplus\lambda_3$ defined by $Q(x_1,x_2,x_3):=E(x_1,x_2)+E(x_2,x_3)+E(x_3,x_1)$, and sgn means the signature of the quadratic form Q, i.e., {the number of positive eigenvalues of Q}-{the number of negative eigenvalues of Q}. We call τ thus defined the *Maslov's index* of $\lambda_1,\lambda_2,\lambda_3$.

Let $\mathrm{LGrass}(T^*M)$ be the set $\cup_{p\in T^*M}\mathrm{LGrass}(T_p(T^*M))$. Then $\mathrm{LGrass}(T^*M)$ is a fiber bundle whose base space is T^*M with the fiber $\mathrm{LGrass}(T_p(T^*M))$ at p. Then the restriction $\mathrm{LGrass}(T^*M)|_{\Lambda_{i\mathbb{R}}'}$ is a fiber bundle on the non-singular Lagrangian subvariety $\Lambda_{i\mathbb{R}}'$. The two sections of $\mathrm{LGrass}(T^*M)|_{\Lambda_{i\mathbb{R}}'}$:

$$\lambda_M;\ p \longmapsto \lambda_M(p):=T_p(\pi^{-1}\pi(p)),$$
$$\lambda_{\Lambda_{i\mathbb{R}}};\ p \longmapsto \lambda_{\Lambda_{i\mathbb{R}}}(p):=T_p\Lambda_{i\mathbb{R}},$$

with $p \in \Lambda'_{i\mathbb{R}}$ are continuous sections. Let μ be another continuous section of $\mathrm{LGrass}(T^*M)|_{\Lambda'_{i\mathbb{R}}}$ satisfying $\mu(p) \cap \lambda_M(p) = \{0\}$ and $\mu(p) \cap \lambda_{\Lambda_{i\mathbb{R}}}(p) = \{0\}$ for each $p \in \Lambda'_{i\mathbb{R}}$. Then we have the following theorem

$\mu(p) \cap \lambda_{\Lambda_{i\mathbb{R}}}(p) = \{0\}$ for each $p \in \Lambda'_{i\mathbb{R}}$. Then we have the following theorem

Theorem 3.3. *Let $\Lambda_{i\mathbb{C}}$ be a Lagrangian subvariety in* $\mathrm{ch}(\mathfrak{M})$ *defined above and let $N_{\mathbb{R}}$ be a subset of $\Lambda'_{i\mathbb{R}}$ consisting of all points in regular position with respect to $\Lambda_{i\mathbb{C}}$. Then*

(3.4) $\sigma_{\Lambda_{i\mathbb{R}}}(F(u)) \times \exp(\frac{\pi}{4}\sqrt{-1}\,\tau(\lambda_M, \lambda_{\Lambda_{i\mathbb{R}}}, \mu))\ |_{\Lambda'_{i\mathbb{R}} - N_{\mathbb{R}}}$

can be extended to the whole $\Lambda'_{i\mathbb{R}}$ and it is a real analytic section on $\Lambda'_{i\mathbb{R}}$.

This is a fundamental property of the real principal symbol and has already been pointed out by Kashiwara in the case of simple microfunctions.

Definition 3.4. (real principal symbols at points in non-regular position) We define the value of $\sigma_{\Lambda'_{i\mathbb{R}}}(F(u))$ on $N_{\mathbb{R}}$ to be the one such that (3.4) is real analytic.

Thus $F(u)$ has been defined as a section on $\Lambda'_{i\mathbb{R}}$ on the whole $\Lambda'_{i\mathbb{R}}$, which may not be continuous at the points in $N_{\mathbb{R}}$. Such extension does not depend on the choice of μ.

Lastly we shall give two theorems.

Theorem 3.5. *Let* F_1 *and* F_2 *be two local sections of* $\mathcal{H}om_{\mathcal{E}_X}(\mathfrak{M}^{\mathcal{E}},\mathcal{C}_M)$ *defined near a point* p *in* $ch(\mathfrak{M})_{reg}$. *If their real principal symbols* $\sigma(F_1(u))$ *and* $\sigma(F_2(u))$ *coincide with each other for any section* u *of* $\mathfrak{M}^{\mathcal{E}}$, *then* F_1 *and* F_2 *are the same section.*

Corollary 3.6. *Let* F_1 *and* F_2 *be two sections of* $\mathcal{H}om_{\mathcal{D}_X}(\mathfrak{M},\mathcal{B}_M)$. *Let* p *be a point in* $ch(\mathfrak{M})_{reg}$ *and let* u *be a local section of* $\mathfrak{M}^{\mathcal{E}}$ *near* p. *If the real principal symbols* $\sigma(sp(F_1(u)))$ *and* $\sigma(sp(F_2(u)))$ *coincide with each other at every point* $p\in ch(\mathfrak{M})_{reg}$ *and for any section* u *of* $\mathfrak{M}^{\mathcal{E}}$, *then* F_1 *and* F_2 *coincide with each other.*

These two theorems follows directly from Theorem 2.1.

References

[Kashiwara1] Kashiwara,M., Microlocal calculus of simple microfunctions, in "Complex Analysis and Algebraic Geometry" (Iwanami and Princeton UP), ed. by Baily,W. and Shiota,T.,(1977),369-374.

[Kashiwara2] Kashiwara,M., Micro-local analysis, in Proceedings of International Congress of Mathematics, 1978, ed. by Olli Lehto, Vol.1, (1980), 139-150.

[Kashiwara3] Kashiwara,M., Systems of microdifferential equations (Progress in Math. Vol.34), Boston, Birkhauser, (1984).

[Kashiwara-Kawai] Kashiwara,M. and Kawai,T., On holonomic systems of microdifferential equations III, Publ. RIMS, Kyoto Univ.,**17**,(1981), 813-979.

[Muro1] Muro,M., On Uniqueness of hyperfunction solutions of a holonomic systems, to appear in *Ark. Mat.*,**26** (1988).

[Muro2] Muro,M.; Singular invariant tempered distributions on regular prehomogeneous vector spaces, to appear in *J. of Funct. Anal.* ,**76**, (1988)

[Sato-Kashiwara-Kimura-Oshima] Micro-local analysis of prehomogeneous vector spaces, *Invent. Math.*,**62**,(1980),117-179.

Department of Math.
Kochi University
Kochi 780 JAPAN